Discrete Mathematics 15 (1976) 141–150.

ALGEBRAIC MODELS FOR TWO-EDGE-CONNECTED GRAPHS

D.J. HARTFIEL and C.J. MAXSON
Department of Mathematics, Texas A & M University, College Station, TX 77843, U.S.A.

Received 3 June 1974

The purpose of this paper is to associate two-edge-connected graphs to a vector space of matrices. The fruitfulness of this association is shown by deriving various graph theoretic results via algebraic arguments on the associated vector space.

The paper is divided into two sections. In the first section, an association between a vector space of matrices and two-edge-connected graphs is developed. The transfer of various properties of graphs to matrices and of matrices to graphs is then considered. In the second section the utility of the association developed in Section 1 is indicated by showing how the algebraic representation may be used, as a tool, to deduce rather interesting results concerning the structure of the associated graphs.

1. Algebraic representation of two-edge-connected graphs

Consider n a fixed positive integer. Our work concerns the following sets:

$$G_n = \{G\colon G \text{ is the union of disjoint two-edge-connected graphs, without loops, on } n \text{ vertices labeled one through } n\}\,,$$

$$M_n = \{A\colon A \text{ is a matrix of order } n\}\,,$$

$$\Lambda_n = \{A\colon A \in M_n \text{ and } r_i(A) = \sum_k a_{ik} = \sum_k a_{ki} = c_i(A) \text{ for } i = 1, 2, \dots, n\}\,,$$

$$T_n = \{A\colon A \in \Lambda_n \text{ and } A \text{ is upper triangular}\}\,.$$

The linear mappings on M_n which are of concern are as follows:

$$\mathrm{up}(A) = B, \quad \text{where } b_{ij} = \begin{cases} a_{ij} & \text{if } i < j\,, \\ 0 & \text{otherwise}\,; \end{cases}$$

$$l(A) = B, \quad \text{where } b_{ij} = \begin{cases} a_{ij} & \text{if } i > j\,, \\ 0 & \text{otherwise}\;; \end{cases}$$

$$\mu(A) = \mathrm{up}(A - A^{\mathrm{t}})\,,$$

$$i_P(A) = P^{\mathrm{t}}AP, \quad \text{where } P \in M_n \text{ is a permutation matrix}\,.$$

The behavior of μ and i_P on M_n is developed in the sequel. The significance of these results becomes apparent as the work progresses.

Lemma 1.1. *For any* $A \in M_n$, $\mu(A) - \mu(A)^{\mathrm{t}} = A - A^{\mathrm{t}}$.

Proof. Note that $l(A - A^{\mathrm{t}}) = -[\mathrm{up}(A - A^{\mathrm{t}})]^{\mathrm{t}} = -\mu(A)^{\mathrm{t}}$. Thus

$$\mu(A) - \mu(A)^{\mathrm{t}} = \mathrm{up}(A - A^{\mathrm{t}}) + l(A - A^{\mathrm{t}}) = A - A^{\mathrm{t}}.$$

Theorem 1.2. *The following diagram commutes.*

$$\begin{tikzcd} M_n \arrow[r, "\mu"] \arrow[d, "i_P"] & M_n \arrow[d, "\mu i_P"] \\ M_n \arrow[r, "\mu"] & M_n \end{tikzcd}$$

Proof. The proof follows from the calculation,

$$\begin{aligned} \mu i_P \mu(A) &= \mu[P^{\mathrm{t}}\mu(A)P = \mathrm{up}[P^{\mathrm{t}}\mu(A)P - (P^{\mathrm{t}}\mu(A)P)^{\mathrm{t}}] \\ &= \mathrm{up}[P^{\mathrm{t}}(\mu(A) - \mu(A)^{\mathrm{t}})P] = \mathrm{up}[P^{\mathrm{t}}AP - P^{\mathrm{t}}A^{\mathrm{t}}P] \\ &= \mu(P^{\mathrm{t}}AP) = \mu i_P(A). \end{aligned}$$

As a direct consequence of this theorem, two corollaries are immediate.

Corollary 1.3. *The following diagram commutes.*

$$\begin{tikzcd} \Lambda_n \arrow[r, "\mu"] \arrow[d, "i_p"] & T_n \arrow[d, "\mu i_p"] \\ \Lambda_n \arrow[r, "\mu"] & T_n \end{tikzcd}$$

Corollary 1.4. *For each $A \in M_n$, $\mu(A) = \mu i_p \mu i_{p^t}(A)$.*

For each $A \in M_n$, denote by $\|A\|$ the graph on n vertices labeled $1, 2, \ldots, n$, so that vertex i and vertex j share an edge if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$ where $i \neq j$. With this notation, the interpretation of Corollary 1.3 is as follows. For any $A \in M_n$, $\|A\|$ and $\|i_p(A)\|$ differ only in that their vertices are labeled differently. Similarly for $A \in T_n$, $\|A\|$ and $\|\mu i_p(A)\|$ differ only in that their vertices are labeled differently. Hence i_p on M_n and μi_p on T_n, in terms of associated graphs, may be considered as "relabeling" maps. These maps allow us to relabel the vertices of any graph so that the appearence of particular subgraphs in the associated matrix, may be specified. For example, we may desire the particular subgraph to appear in the upper left corner of the associated matrix.

The vector space of matrices which is of paramount importance in this work is T_n. Since T_n is a special vector space of matrices, its associated graphs form a special class of graphs. In fact, our next result links the vector space T_n with the graphs of G_n.

Theorem 1.5. *$G \in G_n$ if and only if there is an $A \in T_n$ so that $\|A\| = G$.*

Proof. Suppose $G \in G_n$. Then, as is easily seen, each edge of G is on an elementary cycle of G. Suppose this elementary cycle has edges (i_1, i_2), $(i_2, i_3), \ldots, (i_s, i_1)$. Let B be the $(0, 1)$-matrix with $b_{ij} = 1 \Leftrightarrow i = i_t, j = i_{t+1}$ (t mod s). Then $B \in \Lambda_n$. Suppose now that G has r edges. For the kth edge, let B_k be defined as above. Consider

$$B = \sum_k 2^k B_k \in \Lambda_n \quad \text{and} \quad \|B\| = G\,.$$

Suppose $b_{ij} = b_{ji} > 0$. Then

$$\sum_k 2^k b_{ij}^{(k)} = \sum_k 2^k b_{ji}^{(k)}$$

which may be written as

$$\sum_{b_{ij}^{(k)}>0} 2^k b_{ij}^{(k)} = \sum_{b_{ji}^{(k)}>0} 2^k b_{ji}^{(k)} .$$

Let $k_0 = \min_{b_{ij}^{(k)}>0,\, b_{ji}^{(k)}>0} k$. Suppose, without loss of generality, that $b_{ij}^{(k_0)} > 0$. Hence

$$\sum_{b_{ij}^{(k)}>0} 2^{k-k_0} b_{ij}^{(k)} = \sum_{b_{ji}^{(k)}>0} 2^{k-k_0} b_{ji}^{(k)} .$$

But now, as $b_{ij}^{(k)} > 0$ only if $b_{ji}^{(k)} = 0$ for each k, we have an odd integer equal to an even integer. Thus if $b_{ij} \neq 0$ then $b_{ij} \neq b_{ji}$. Finally, if we set $\mu(B) = A$ then $A \in T_n$ and $\|A\| = \|B\| = G$.

Conversely, suppose $A \in T_n$. If $\|A\| \notin G_n$ then there is a partition of the vertices of $\|A\|$ into sets, say V_1 and V_2, such that there is precisely one edge of $\|A\|$ having a vertex in V_1 and a vertex in V_2.

Relabel the vertices in V_1 as $1, 2, \ldots, k$ and those in V_2 as $k+1, k+2, \ldots, n$, yielding the graph $G' \in G_n$. Then there is a permutation matrix P so that $\|\mu i_P(A)\| = G'$. Thus setting $A' = \mu i_P(A)$ we have that $\|A'\| = G'$. Now,

$$A' = \begin{pmatrix} A_1 & E_1 \\ 0 & A_2 \end{pmatrix}$$

where A_1 is of order k and E_1 has precisely one nonzero entry. Hence

$$\sum_{i=1}^{k} r_i(A') = \sum_{i=1}^{k} c_i(A') \quad \text{as } A' \in T_n .$$

But

$$\sum_{i=1}^{k} r_i(A') \neq \sum_{i=1}^{k} c_i(A'),$$

by direct calculation. Thus we have a contradiction from which we conclude the theorem.

If $G \in G_n$ and $A \in T_n$ so that $\|A\| = G$ we call A an algebraic model of G. Of course, with each G there are many such models. This can be illuminated by defining a relation on T_n as follows:

Let $R \subset T_n \times T_n$ so that $(A, B) \in R$ if and only if $\|A\| = \|B\|$. It is easily shown that R is an equivalence relation on T_n. We denote the equivalence class to which A belongs as $[\|A\|]$. Thus $[\|A\|]$ consists of the set of all models of $\|A\|$.

Having established a relationship between G_n and T_n we now proceed to identify some of the structure of $G \in G_n$ with the structure of its models in T_n. In particular, we shall be interested in identifying the paths and cycles of G in its associated models.

If $A \in M_n$, then any sequence of entries of A, of type

(a) $a_{i_1 j_1}, a_{i_2 j_1}, a_{i_2 j_2}, a_{i_3 j_2}, \ldots, a_{i_r j_r}$,

(b) $a_{i_1 j_1}, a_{i_2 j_1}, a_{i_2 j_2}, a_{i_3 j_2}, \ldots, a_{i_r j_r}, a_{i_{r+1} j_r}$,

(c) $a_{i_1 j_1}, a_{i_1 j_2}, a_{i_2 j_2}, a_{i_2 j_3}, \ldots, a_{i_r j_r}$,

(d) $a_{i_1 j_1}, a_{i_1 j_2}, a_{i_2 j_2}, a_{i_2 j_3}, \ldots, a_{i_r j_{r+1}}$,

where $a_{ij} = 0$ in any sequence only if $i = j$, is called a chain in A. The chain of type (a) is closed if $i_1 = i_r$ and $j_1 = j_r$. A chain of type (c) is closed if $i_1 = i_r$ and $j_1 = j_r$. Loosely speaking, a chain in A is a sequence of entries in A obtained by alternately making row and column moves through nonzero entries, except for main diagonal entries, of A. A chain is closed if it is possible to return to the initial position in an even number of such moves.

Lemma 1.6. *Let $A \in T_n$. Then*

(1) *A has a chain of*

type (a) $\Leftrightarrow$ $\|A\|$ *has a path from* i_1 *to* i_r,

type (b) $\Leftrightarrow$ $\|A\|$ *has a path from* i_1 *to* i_{r+1},

type (c) $\Leftrightarrow$ $\|A\|$ *has a path from* j_1 *to* i_r,

type (d) $\Leftrightarrow$ $\|A\|$ *has a path from* j_1 *to* j_{r+1};

(2) *A has a closed chain* $\Leftrightarrow$ $\|A\|$ *has a cycle.*

Proof. The result follows by translating the matrix language into the language of graphs and visa versa, being careful to add diagonal entries when necessary.

The identification of paths and cycles of a graph $G \in G_n$, in an algebraic model of G, is much simplified by the following theorem.

Theorem 1.7. *$G \in G_n$ has an elementary path of length $k-1$ if and only if there is a permutation matrix P so that for any $A \in [G] \subset T_n$, $[\|\mu i_P(A)\|] = [\|B\|]$, where B has a chain consisting of entries*

$$b_{12}, b_{22}, b_{23}, \dots, b_{k-1\,k-1}, b_{k-1\,k}.$$

$G \in G_n$ has an elementary cycle of length k if and only if there is a permutation matrix P so that for any $A \in [G] \subset T_n$, $[\|\mu i_P(A)\|] = [\|B\|]$, where B has a chain consisting of entries

$$b_{12}, b_{22}, b_{23}, \dots, b_{k-1\,k-1}, b_{k-1\,k}, b_{1\,k}.$$

Proof. First suppose G has an elementary path of length $k-1$. Relabel the vertices of G yielding G' so that in G', the vertices in the path are labeled sequencially by $1, 2, \dots, k$. Now if $B \in T_n$ with $\|B\| = G'$ then $b_{12}, b_{22}, b_{23}, \dots, b_{k-1\,k-1}, b_{k-1\,k}$, is a chain in B. Since G' is obtained from G by relabeling, there is a permutation P and $A' \in [G]$ such that $\mu i_P(A') = B$. Thus for each $A \in [G]$, $\|\mu i_P(A)\| = \|A'\| = \|B\|$.

Conversely, suppose there is a permutation matrix P so that for any $A \in [G] \subset T_n$, $[\|\mu i_P(A)\|] = [\|B\|]$ where B has a chain consisting of entries

$$b_{12}, b_{22}, b_{23}, \dots, b_{k-1\,k-1}, b_{k-1\,k}, b_{1\,k}.$$

Then $\|B\|$ has an elementary path of length $k-1$. As $\|\mu i_P(A)\| = \|B\|$, it follows from Lemma 1.6 that G has an elementary cycle of length $k-1$.

The proof of the second statement is similar in nature and left to the reader.

Utilizing the relabeling map μi_P it is also possible to determine if a graph $G \in G_n$ is connected.

Theorem 1.8. *$G \in G_n$ is not connected if and only if there is a permutation matrix P so that for any $A \in [G] \subset T_n$, $[\|\mu i_P(A)\|] = [\|B\|]$ where $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ and $B_1 \in T_k$, $B_2 \in T_{n-k}$ for some k, $0 < k < n$.*

Proof. If G is not connected, then there is a partition of the vertices of G into sets, say V_1 and V_2, so that there is no edge of G having a vertex in V_1 and a vertex in V_2. Relabel the vertices of B_1 as $1, 2, \ldots, k$ and those of V_2 as $k+1, \ldots, n$, yielding $G' \in G_n$. Then there is a permutation matrix P so that for any $A \in [G] \subset T_n$, $\|\mu i_P(A)\| = G'$. Thus setting $B = \mu i_P(A)$ it follows that $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ with $B_1 \in T_k$ and $B_2 \in T_{n-k}$ for $0 < k < n$.

Conversely, if there is a permutation matrix P so that $\|\mu i_P(A)\| = \|B\|$ and $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ with $B_1 \in T_k$ and $B_2 \in T_{n-k}$ then $\|B\|$ is not connected and hence neither is $\|A\| = G$.

The foregoing results indicate how various properties of a graph $G \in G_n$ may be identified in an algebraic model $A \in T_n$ of G. This then establishes a clear relationship between the sets G_n and T_n. Our goal now is to use this relationship, and the fact that T_n is a vector space, to derive properties about the graphs in G_n.

2. Applications of T_n to G_n

The purpose of this section is to show how T_n may be used to deduce truths concerning G_n. The first part is concerned with minimally two-edge-connected graphs.

Suppose $G(X, T) \in G_n$ is connected with the property that if e is any edge of G, $G'(X, T-e) \notin G_n$. We then call $G(X, T)$ minimally two-edge-connected. Algebraic models of two-edge-connected graphs are very beneficial in the study of minimally two-edge-connected graph. A characterization of these models is the substance of our next theorem.

Theorem 2.1. *G is not minimally two-edge-connected if and only if G has a model with a unique nonzero entry of minimum modulus.*

Proof. Suppose G is two-edge-connected. Let e be an edge of G so that $G'(X, T-e)$ is two-edge-connected. Let $A \in T_n$ be a model of G' and $B \in T_n$ a model of G. Now consider $C = A + \lambda B$. For sufficiently small λ, C is a model of G with a unique nonzero entry of minimum modulus. In particular, this entry in C corresponds to e in G.

Conversely, suppose G has a model in T_n with a unique nonzero entry of minimum modulus. Suppose this entry corresponds to edge e of G.

As G is two-edge-connected, e is on an elementary cycle. This cycle is two-edge-connected and hence has a model B in T_n. Note that $b_{ij} \neq 0$ implies $a_{ij} \neq 0$. Also note that the nonzero entries of B have the same modulus. Thus, if a_{pq} corresponds to e, picking λ so that $a_{pq} + \lambda b_{pg} = 0$ implies that C is a model of $G'(X, T-e)$. As G was two-edge-connected, so is G'. Hence G is not minimally two-edge-connected.

The use of such a theorem as above is exemplified by the following result.

Theorem 2.2. *If $G(X, T)$ is minimally two-edge-connected with a triangle formed by edges e_1, e_2 and e_3, then $G''(X, T-e_1-e_2-e_3)$ is the disjoint union of three minimally two-edge-connected graphs.*

Proof. Relabel the vertices of G, yielding G', so that the vertices involved in the triangle are labeled one, two and three. Now if $A \in T_n$ is a model of G', $a_{12} \neq 0$, $a_{13} \neq 0$ and $a_{23} \neq 0$. Let E be an $n \times n$ matrix so that

$$e_{ij} = \begin{cases} 1 & \text{if } i=1, j=2 \text{ and } i=2, j=3\,, \\ -1 & \text{if } i=1, j=3\,, \\ 0 & \text{otherwise.} \end{cases}$$

Consider $C = A + \lambda E$. Pick λ so that $\lambda + a_{12} = 0$. As G' is minimally two-edge-connected, $-\lambda + a_{13} = 0$ and $\lambda + a_{23} = 0$. Thus C is a model for $G''(X, T-e_1-e_2-e_3)$ and hence G'' is a disjoint union of two-edge-connected graphs. That there are precisely three minimally two-edge-connected graphs which comprise G'' is an elementary combinatorial argument and left to the reader.

Another result concerning the structure of two-edge-connected graphs is as follows.

Theorem 2.3. *If $G(X, T)$ is two-edge-connected and $e_1, e_2, \ldots, e_s$ are the edges of some elementary cycle of G, then there is some non-empty subcollection of these edges whose removal from T yields a graph G'' which is a disjoint union of two-edge-connected graphs.*

Proof. Relabel the vertices of G yielding G', so that vertices labeled i and $i+1$ are end points of e_i. Pick any model of G' say $A \in T_n$. Let $E \in T_n$ be a model of the subgraph of G' formed from the vertices of G' and edges

$e_1, e_2, \ldots, e_s$. Consider $C = A + \lambda E$. Then $C \in T_n$. Thus a prudent choice of λ yields the result.

This theorem indicates that two-edge-connected graphs are constructed by covering the vertices with elementary cycles in such a manner that the constructed graph is connected. In the context of minimally two-edge-connected graphs, the theorem indicates that any such graph is composed of a disjoint union of minimally two-edge-connected graphs on fewer vertices with this disjoint union connected by an elementary cycle.

As a final application for the use of the association between T_n and G_n, we present a theorem originally given by Robbins [3].

Theorem 2.4. *The edges of a graph G can be oriented to form a strongly connected digraph if and only if G is two-edge-connected.*

Proof. Suppose the edges of G can be oriented to form a strongly connected graph, say $\vec{G}$. Then by the theory established in [2], $\vec{G}$ has a model $M \in \Lambda_n$ with $a_{ij} \neq 0$ implying that $a_{ji} = 0$. Hence $\mu(M) \in T_n$ and so $G = \|\mu(M)\| \in G_n$. As $\vec{G}$ is strongly connected, G is connected and hence G is two-edge-connected.

Conversely, suppose G is two-edge-connected. Let $M \in T_n$ so that $\|M\| = G$. Now in M, if $m_{ij} < 0$, replace m_{ij} by 0 and m_{ji} by $-m_{ij}$ yielding $M^+ \geqslant 0$ and $M^+ \in \Lambda_n$. Thus, again by the work in [2], M^+ is a model for a strongly connected graph.

We remark that this theorem actually indicates an orientation scheme to convert a two-edge-connected graph into a strongly connected graph. For this, if $M \in T_n$ with $\|M\| = G$ then orient the edge from vertex i to vertex j if $m_{ij} > 0$ and from vertex j to vertex i if $m_{ij} < 0$. This orientation provides the strongly connected digraph.

It is also interesting to note that the models of strongly connected digraphs in [2] and the models of two-edge-connected graphs of this paper provide an algebraic comparison of the two types of graphs, i.e., the nonnegative cone of Λ_n provides the models of strongly connected graphs while T_n provides the models of two-edge-connected graphs. Some work relating the connectivity of digraphs to connectivity of graphs has been done by Nash-Williams [4, p. 133], in which this type of relationship was a major point of interest.

References

[1] C. Berge, The Theory of Graphs (Dunod, Paris, 1958; English transl., Wiley, New York, 1962).
[2] D.J. Hartfiel and J.W. Spellmann, A role for doubly stochastic matrices in graph theory, Proc. Am. Math. Soc. 36 (1972) 389–394.
[3] H.E. Robbins, A theorem on graphs, with an application to a problem on traffic control, Am. Math. Monthly 46 (1936) 281–283.
[4] W.T. Tutte, Recent Progress in Combinatorics (Academic Press, New York, 1969).